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FREE LATERAL VIBRATION OF AN AXIALLY CREEPING BEAM-COLUMN UNDER INITIAL AXIAL COMPRESSION*

by

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ABSTRACT

The free lateral vibration of a nonlinear viscoelastic beam-column subjected to an initial compressive axial load is considered. The constitutive law is formulated with a linear elastic term and with power functions of stress in the transient and steady creep terms, and is of the nonlinear generalized Kelvin type.

By assuming that the stress caused by the oscillation is of much smaller magnitude than the initial stress, the problem is linearized. The problem is analyzed for five special viscoelastic models using small deformation theory, and numerical results are discussed for a stainless steel alloy.

1. INTRODUCTION

As noted by Hoff [1], creep in metals must have a damping effect on mechanical vibration, since it absorbs energy. If one is considering a vibration problem in a temperature range for which creep may be present, the damping due to creep should be included in the analysis. Cozzarelli, Wu and Tang[2] considered a beam constructed of a material which may be represented by a nonlinear Maxwell-Kelvin model, and which was subjected to a free lateral vibration while under initial axial tension. In this paper we extend their analysis in two ways; the first of which is to employ a more general viscoelastic law of the nonlinear generalized Kelvin type in order to provide greater flexibility in the fitting of experimental data. And secondly we extend the type of loading to include the case of initial axial compression. The growth or decay of damped vibration under compression is a problem of practical importance in studies of structural reliability and safety, and thus we give particular attention to this aspect of our study. Creep collapse of a fuel element in a nuclear reactor is an example of such problems which is of current interest.

The particular case considered here is a nonlinear viscoelastic beam-column which is simply supported and under an initial axial compressive load. The beam-column is assumed to be initially straight and of an isothermal homogeneous medium with a rectangular cross-section. By assuming that the initial stress increment is of much higher magnitude than the stress increment caused by the oscillation of the beam, the problem is linearized in a manner analogous to that in ref. 2. The incremental approach to the theory of dynamic creep stability in columns has been developed in a number of papers; of these we mention in particular Rabotnov and Shesterikov[3,4], Jahsman and Field[5], Jahsman[6], and Distéfano and Sackman[7,8]. The present development more closely parallels that in ref. 7-8, since it is based on a nonlinear viscoelastic relation rather than on a mechanical equation of state as in ref. 3-6. Our specific goal is to demonstrate in detail how for actual values of the creep parameters one may readily construct the complete incremental displacement solution for rather general constitutive relations primarily by means of simple hand calculations.

The governing displacement equation is derived in Section 2 using small deformation theory, and the solution of this equation for a nonlinear Maxwell-Kelvin-Kelvin (or simpler) material is discussed in Section 3. This letter section employs the property of physically small damping due to creep to develop simple but accurate estimates to the roots of the characteristic equations. Numerical results for a stainless steel are considered in Section 4, where we examine the effects of steady creep, slow transient creep and rapid transient creep.

2. DERIVATION OF THE GOVERNING EQUATION

2.1 Constitutive Law

A one-dimensional constitutive law of the nonlinear generalized Kelvin type, similar to the law discussed in detail and used in ref. 9, will be employed here in its integral form

$$\epsilon(T) = \left[\frac{|\sigma(T)|}{E} + \int_0^T \left|\frac{\sigma(T')}{\lambda}\right|^n dT' + \sum_{i=1}^{N} \frac{e^{-T/\tau_i}}{\tau_i} \int_0^T \left|\frac{\sigma(T')}{\mu_i}\right|^{q_i} e^{T'/\tau_i}\right] \mathrm{sgn}\,\sigma(T \tag{1}$$

Here, T, $\epsilon(T)$ and $\sigma(T)$ represent time, strain and stress respectively and E is the modulus of elasticity; λ is the steady creep parameter and n is the steady creep power; τ_i are retardation times, μ_i are transient creep parameters and q_i are transient creep powers; and $\mathrm{sgn}\,\sigma(T)$ is the usual signum function. By setting N=1 in eq. (1) we arrive at the constitutive equation used in ref. 2, except that here the use of the signum function enables us to consider negative stresses.

Consider a material with an initial constant stress σ_o (positive or negative), subjected to a small disturbance at $T=T_o$. This disturbance is translated into a small additional increment of stress, i.e.

$$\sigma(T) = \sigma_o + \varepsilon\tilde{\sigma}(T) \tag{2}$$

where $\tilde{\sigma}(T)$ is the stress increment and ε is a small quantity. It then follows that

$$|\sigma|^n \,\mathrm{sgn}\,\sigma = |\sigma_o|^n \,\mathrm{sgn}\,\sigma_o + n\,|\sigma_o|^{n-1}\varepsilon\tilde{\sigma} + \dots \tag{3a}$$

$$|\sigma|^{q_i} \,\mathrm{sgn}\,\sigma = |\sigma_o|^{q_i} \,\mathrm{sgn}\,\sigma_o + q_i\,|\sigma_o|^{q_i-1}\varepsilon\tilde{\sigma} + \dots \tag{3b}$$

Also, as a result of the disturbance the strain assumes the form

$$\epsilon(T) = \epsilon_o(T) + \varepsilon\tilde{\epsilon}(T) + \dots \tag{4}$$

Substituting eqs. (2)-(4) into eq. (1), subtracting the relation for the initial state, and retaining terms of order ε, we obtain the following linear constitutive relation in the increments:

$$\tilde{\epsilon}(T) = \frac{\tilde{\sigma}(T)}{E} + \int_{T_o}^{T} \frac{n\,|\sigma_o|^{n-1}}{\lambda^n}\tilde{\sigma}(T')\,dT' + \sum_{i=1}^{N} \frac{e^{-T/\tau_i}}{\tau_i}\int_{T_o}^{T} \frac{q_i\,|\sigma_o|^{q_i-1}}{\mu_i^{q_i}} \cdot \tilde{\sigma}(T')\,e^{T'/\tau_i}\,dT' \qquad T > T_o \tag{5}$$

Eq. (5) may also be expressed as the memory integral

$$\tilde{\epsilon}(T) = \int_{T_o^-}^{T} \left[\frac{1}{E} + \frac{n\,|\sigma_o|^{n-1}}{\lambda^n}(T-T') + \sum_{i=1}^{N} \frac{q_i\,|\sigma_o|^{q_i-1}}{\mu_i^{q_i}} \left(1-e^{-(T-T')/\tau_i} \right) \right]$$

$$\cdot \frac{\partial \tilde{\sigma}(T')}{\partial T'} dT' \qquad T \geq T_o^- \tag{6}$$

where we have used $\tilde{\sigma}(T_o^-) = 0$.

2.2 Governing Equation for Lateral Vibration of an Axially Creeping Beam-Column

As a specific example we consider the vibration of an initially straight Euler-Bernoulli beam-column, which is subjected at $T=0$ to a prescribed axial force P_o (tensile or compressive) followed later by a small lateral disturbance at $T=T_o$. Using the perturbation technique employed in obtaining eq. (6), the classical strain displacement and equilibrium relations for such a beam-column yield the following relations in the first order terms:

$$\tilde{\epsilon}(X,T) = -Y \frac{\partial^2 \tilde{V}(X,T)}{\partial X^2} \tag{7a}$$

$$\frac{\partial^2 \tilde{M}(X,T)}{\partial X^2} = P_o \frac{\partial^2 \tilde{V}(X,T)}{\partial X^2} - \rho \frac{\partial^2 \tilde{V}(X,T)}{\partial T^2} + W(X,T) \tag{7b}$$

$$T \geq T_o^-$$

In eqs. (7), X is the neutral axis of the beam-column which vibrates in the XY plane, $V(X,T)$ is the lateral displacement, $\tilde{M} = -\int_A \tilde{\sigma} Y dA$ is the bending moment for a beam-column with cross-section A, ρ is the linear density, and W is the lateral load which will be used to determine the initial conditions of the free vibration.

For convenience, we introduce the nondimensional quantities

$$v = \frac{\tilde{V}}{V^*}, \quad x = \frac{X}{L}, \quad t = \frac{T-T_o}{T^*}, \quad w = \frac{T^{*2}}{\rho V^*} W \tag{8}$$

Here, V^* is some convenient constant reference displacement, L is the length of the beam, T^* is a convenient factor which has the dimension of time, and v, x, t and w are the non-dimensional lateral displacement, axial coordinate, time and lateral load, respectively.

Combining eqs. (6), (7) and (8) we finally obtain the governing equation

$$\frac{\partial^4 v}{\partial x^4} = \int_0^t \left[a \frac{\partial^3 v}{\partial x^2 \partial t'} - b \frac{\partial^3 v}{\partial t'^3} + b \frac{\partial w}{\partial t'} \right]$$

$$\cdot \left[1 + c(t-t') + \sum_{i=1}^{N} d_i \left(1 - e^{-j_i(t-t')} \right) \right] dt' \tag{9}$$

$$t \geq 0^-$$

where

$$a = \frac{P_o L^2}{EI};\ b = \frac{\rho L^4}{EIT^{*2}};\ c = \frac{En|P_o/A|^{n-1} T^*}{\lambda^n};\ d_i = \frac{Eq_i |P_o/A|^{q_i - 1} T^*}{\tau_i \mu_i^{q_i}};$$

$$j_i = \frac{T^*}{\tau_i} \qquad i = 1, \ldots, N \tag{10}$$

and where the second moment of area is $I = \int_A Y^2 dA$. Note that the nondimensional force a has the same sign as P_o, whereas b, c, d_i and j_i are all positive nondimensional material parameters. Also note that as a result of material nonlinearity the parameters c and d_i depend on $|P_o|$.

In Section 3, eq. (9) will be analyzed for five different models. These models are, in order of decreasing complexity, *nonlinear* Maxwell-Kelvin-Kelvin $[\tau_i \to \infty$ (d_i and $j_i \to 0$) for $i \geq 3]$, nonlinear Maxwell-Kelvin $[\tau_i \to \infty$ (d_i and $j_i \to 0$) for $i \geq 2]$, nonlinear standard solid $[\tau_i \to \infty$ (d_i and $j_i \to 0$) for $i \geq 2$ and $\lambda \to \infty$ $(c \to 0)]$, nonlinear Maxwell $[\tau_i \to \infty$ (d_i and $j_i \to 0$) for $i \geq 1]$ and linear elastic $[\tau_i \to \infty$ (d_i and $j_i \to 0$) for $i \geq 1$ and $\lambda \to \infty (c \to 0)]$. The differential form of eq. (9) for the nonlinear Maxwell-Kelvin-Kelvin (M-K-K) model is obtained via successive differentiation as

$$\frac{1}{b} \frac{\partial^7 v}{\partial x^4 \partial t^3} + \frac{j_1 + j_2}{b} \frac{\partial^6 v}{\partial x^4 \partial t^2} + \frac{j_1 j_2}{b} \frac{\partial^5 v}{\partial x^4 \partial t} = \frac{a}{b} \frac{\partial^5 v}{\partial x^2 \partial t^3} - \frac{\partial^5 v}{\partial t^5} + \frac{\partial^3 w}{\partial t^3}$$

$$+ (c + d_1 + d_2 + j_1 + j_2) \left(\frac{a}{b} \frac{\partial^4 v}{\partial x^2 \partial t^2} - \frac{\partial^4 v}{\partial t^4} + \frac{\partial^2 w}{\partial t^2} \right)$$

$$+ \left[j_1(c + d_2) + j_2(c + d_1) + j_1 j_2 \right] \left(\frac{a}{b} \frac{\partial^3 v}{\partial x^2 \partial t} - \frac{\partial^3 v}{\partial t^3} + \frac{\partial w}{\partial t} \right)$$

$$+ c\, j_1 j_2 \left(\frac{a}{b} \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial t^2} - w \right) \qquad t \geq 0^- \tag{11}$$

The remaining differential forms for the other special models are easily obtained from the limits listed above.

Next, the nondimensional loading function is expressed in the form

$$w = g(x)\,\eta(t) \tag{12}$$

where $g(x)$ is a function of x, and $\eta(t)$ is a doublet function which excites a free vibration for $t \geqslant 0^+$. For the solution of eq. (11) with forcing function (12), it is convenient to set w=0 in eq. (11) and use initial conditions at $t=0^+$. For these conditions we integrate eq. (11) successively between $t=0^-$ (where all quantities vanish) and $t=0^+$ and obtain

$$v(x,0^+) = g(x) \tag{13 a}$$

$$\frac{\partial v}{\partial t}(x,0^+) = 0 \tag{13 b}$$

$$\frac{\partial^2 v}{\partial t^2}(x,0^+) = -\frac{1}{b}\frac{d^4 g(x)}{dx^4} + \frac{a}{b}\frac{d^2 g(x)}{dx^2} \tag{13 c}$$

$$\frac{\partial^3 v}{\partial t^3}(x,0^+) = \frac{c + d_1 + d_2}{b}\,\frac{d^4 g(x)}{dx^4} \tag{13 d}$$

$$\frac{\partial^4 v}{\partial t^4}(x,0^+) = \frac{1}{b^2}\frac{d^8 g(x)}{dx^8} - \frac{2a}{b^2}\frac{d^6 g(x)}{dx^6} + \frac{1}{b}\left[\frac{a^2}{b} - (d_1 j_1 + d_2 j_2) - (c + d_1 + d_2)^2\right]\frac{d^4 g(x)}{dx^4} \tag{13 e}$$

We see that the beam-column has in effect been deformed laterally $t=0^+$ in the form of $g(x)$ and then released from rest.

3. SOLUTION OF THE GOVERNING EQUATION

3.1 The Characteristic Equation

As previously noted, we shall consider the free vibration of a simply-supported nonlinear Maxwell-Kelvin-Kelvin (M-K-K) beam-column (and simpler). Noting the boundary conditions

$$v(0,t) = v(1,t) = \frac{\partial^2 v(0,t)}{\partial x^2} = \frac{\partial^2 v(1,t)}{\partial x^2} = 0 \tag{14}$$

and the fact that all the space derivatives in governing eq. (11) are of even order, we shall take $g(x)$ of eq. (12) in the convenient form

$$g(x) = \sin(m\pi x) \tag{15}$$

where m indicates the mode of vibration. For m=1 we see from eq. (13 a) and (15) that $v(1/2, 0^+) = 1$, which in turn requires for this case that V^* in eq. (8) equal the initial lateral displacement at mid-span.

A solution to eq. (11) then follows simply as

$$v(x, t) = \bar{T}(t) \sin (m\pi x) \tag{16}$$

where $\bar{T}(t)$ is a function of time. Substituting eq. (16) into eq. (11) with the load w=0, we obtain an ordinary differential equation in $\bar{T}(t)$ as

$$\begin{aligned} &\bar{b}\,\overset{.....}{\bar{T}}(t) + \bar{b}(c + d_1 + d_2 + j_1 + j_2)\overset{....}{\bar{T}}(t) \\ &+ \left[\bar{b}(j_1 c + j_2 c + j_1 j_2 + d_1 j_2 + d_2 j_1) + \bar{a} + 1\right]\dddot{\bar{T}}(t) \\ &+ \left[j_1 j_2 \bar{b} c + \bar{a}(c + d_1 + d_2 + j_1 + j_2) + j_1 + j_2\right]\ddot{\bar{T}}(t) \\ &+ \left[\bar{a}(j_1 c + j_2 c + j_1 j_2 + d_1 j_2 + d_2 j_1) + j_1 j_2\right]\dot{\bar{T}}(t) \\ &+ j_1 j_2\, \bar{a}\, c\, \bar{T}(t) = 0 \end{aligned} \tag{17}$$

where $(\dot{\ })$ indicates d/dt and for mode m

$$\bar{a} = \frac{a}{\pi^2 m^2}; \quad \bar{b} = \frac{b}{\pi^4 m^4} \tag{18}$$

The characteristic equation for the nonlinear M-K-K material is obtained by substituting

$$\bar{T}(t) = e^{\xi t} \tag{19}$$

into eq. (17). From this point on we shall confine our attention to the case of initial compression $(\bar{a} = -|\bar{a}|)$, for which the characteristic equation is given by

$$\begin{aligned} &\bar{b}\xi^5 + \bar{b}(c + d_1 + d_2 + j_1 + j_2)\xi^4 \\ &+ \left[\bar{b}(j_1 c + j_2 c + j_1 j_2 + d_1 j_2 + d_2 j_1) - |\bar{a}| + 1\right]\xi^3 \\ &+ \left[j_1 j_2 \bar{b} c - |\bar{a}|(c + d_1 + d_2 + j_1 + j_2) + j_1 + j_2\right]\xi^2 \\ &+ \left[-|\bar{a}|(j_1 c + j_2 c + j_1 j_2 + d_1 j_2 + d_2 j_1) + j_1 j_2\right]\xi \\ &- j_1 j_2 |\bar{a}| c = \sum_{i=0}^{5} \alpha_i \xi^{5-i} = 0 \end{aligned} \tag{20}$$

where the α_i are the six coefficients in this fifth order equation. The characteristic equation for the simpler models are obtained by allowing the appropriate creep parameters to tend toward zero, as described in Section 2.2. For convenience, the coefficients α_i in the characteristic equations for all of the models are summarized in Table 1. Note that in the summation

$$\sum_{i=0}^{p} \alpha_i \xi^{p-i}$$

the quantity p is the order of the equation, which equals 2, 3, 3, 4 and 5 for the E, M, S-S, M-K and M-K-K materials respectively.

Having obtained the roots to the various characteristic equations, the general solution to eq. (11) for unrepeated roots then follows from eqs. (16) and (19) as

$$v(x, t) = \sum_{i=1}^{p} c_i e^{\xi_i t} \sin(m\pi x) \tag{21}$$

...re the c_i are constants which may be determined from initial con-...itions, eq. (13).

TABLE 1 - Coefficients in Characteristics Equations $\sum_{i=0}^{p} \alpha_i \xi^{p-i}$

	Maxwell - Kelvin - Kelvin (p=5)	Maxwell - Kelvin (p=4)	Standard Solid (p=3)	Maxwell (p=3)	Elastic (p=2)
α_0	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$	$\bar{b}$
α_1	$\bar{b}(c+d_1+d_2+j_1+j_2)$	$\bar{b}(c+d_1+j_1)$	$\bar{b}(d_1+j_1)$	$\bar{b}c$	0
α_2	$\bar{b}(j_1c+j_2c+j_1j_2+d_1j_2+d_2j_1)$ $+1-\lvert\bar{a}\rvert$	$\bar{b}j_1c+1-\lvert\bar{a}\rvert$	$1-\lvert\bar{a}\rvert$	$1-\lvert\bar{a}\rvert$	$1-\lvert\bar{a}\rvert$
α_3	$j_1j_2\bar{b}c+j_1+j_2$ $-\lvert\bar{a}\rvert(c+d_1+d_2+j_1+j_2)$	$j_1-\lvert\bar{a}\rvert(c+d_1+j_1)$	$j_1-\lvert\bar{a}\rvert(d_1+j_1)$	$-\lvert\bar{a}\rvert c$	-
α_4	j_1j_2 $-\lvert\bar{a}\rvert(j_1c+j_2c+j_1j_2+d_1j_2+d_2j_1)$	$-\lvert\bar{a}\rvert j_1c$	-	-	-
α_5	$-j_1j_2c\lvert\bar{a}\rvert$	-	-	-	-

3.2 Discussion of Roots and Solutions for Initial Compression

In this section we examine the nature of the roots to the five characteristic equations listed in Table 1, and also discuss the corresponding solutions in accordance with eq. (21). Beginning with the elastic material we see that the two roots are imaginary for $|\bar{a}|<1$ and real (one positive) for $|\bar{a}|>1$. Thus the solution for $|\bar{a}|>1$ exhibits exponential growth, and we recognize the value $|\bar{a}|=1$ [i.e. $|P_o| = m^2\pi^2 EI/L^2$ via eq. (10) and eq. (18)] as the critical buckling load for an elastic column. On the other hand, the solution for $|\bar{a}|<1$ oscillates with the nondimensional natural frequency

$$\omega = \left(\frac{1-|\bar{a}|}{\bar{b}}\right)^{1/2} \tag{22}$$

which decreases as the initial load increases. Since actual structures are usually designed to operate at loads below this critical load, we shall assume that for all the models

$$|\bar{a}| < 1 \tag{23}$$

Now we consider as a group the four other viscoelastic models listed in Table 1, and remark that the damping due to creep is physically small (see ref. 2). Accordingly, the solutions for all these viscoelastic models will contain two terms which oscillate at essentially the elastic frequency ω as given by eq. (22) and decay slowly with time. Such terms correspond to two roots with negative real parts

$$\xi_{1,2} \approx -\beta \pm i\omega \quad \text{where } \omega \gg \beta \tag{24}$$

which implies that the creep parameters are of small order β when compared with ω. If we substitute eq. (24) with (22) into the characteristic equation for the nonlinear M-K-K material and ignore terms of higher order, we find that the imaginary part is identically satisfied while the real part yields the relation

$$\beta \approx \frac{c + d_1 + d_2}{2(1-|\bar{a}|)} \tag{25}$$

Eq. (25) applies equally well to the nonlinear M-K material (d_2=0), the nonlinear S-S material ($c=d_2=0$), and to the nonlinear M material ($d_1=d_2=0$).

The remaining p-2 roots will now be examined for the various models with the aid of Descartes' rule of signs. We see from Table 1 that in all four models the coefficients α_o and α_1 are positive, and α_2 is also positive due to inequality (23). For the Maxwell material we note the α_3 is always negative, and thus the remaining third root for this material must be positive real, and the total solution contains two decaying oscillatory terms and one exponentially growing term. We may obtain a good estimate for this positive real root by solving the characteristic equation with $\bar{b} = 0$, since this is equivalent to dropping the inertia and the associated oscillatory terms. In so doing we obtain

$$\xi_3 \approx \frac{|\bar{a}|\, c}{1 - |\bar{a}|} \tag{26}$$

For the standard solid material we see that α_3 may be positive or negative, depending on the value of $|\bar{a}|$. Thus, if

$$|\bar{a}| > \bar{a}_c \qquad \text{where } \bar{a}_c = j_1/(d_1 + j_1) < 1 \tag{27}$$

the remaining third root must be positive real, whereas if $|\bar{a}| < \bar{a}_c$ this root is negative real. The estimate to this root, as obtained by setting $\bar{b}=0$, is given by

$$\xi_3 \approx \frac{|\bar{a}|(d_1 + j_1) - j_1}{1 - |\bar{a}|} \tag{28}$$

which exhibits the above mentioned behaviour in regard to sign. In contrast to the M material, the exponential term in the total solution for the S-S material grows in time only if the load is large enough for inequality (27) to be satisfied. Thus for this material there is a second critical load $|P_{oc}|$ defined by $|\bar{a}| = \bar{a}_c$, which upon using equations (10), (18) and (27) yields the nonlinear expression in $|P_{oc}|$

$$\frac{|P_{oc}|\, L^2}{m \pi^2 I} \left[\frac{q_1 \, |P_{oc}/A|^{q_1 - 1}}{\mu_1^{\,q_1}} + \frac{1}{E} \right] = 1 \tag{29}$$

The quantity in brackets is the effective reciprocal elastic modulus of the linear spring in series with the nonlinear Kelvin element (with $\tau_1 = 0$). Note that $|P_{oc}|$ is less than the critical elastic buckling load, and also that due to material nonlinearity the elastic modulus of the Kelvin element depends on $|P_{oc}|$ itself.

Consider next the nonlinear Maxwell-Kelvin material, where there are two roots in addition to the complex pair given by eq. (24). Again examining Table 1, we see that whereas α_4 is always negative, α_3 may be either positive or negative. However, in both cases there is only one variation in sign, and thus there is one positive real and one negative real root. Setting $\bar{b}=0$ in Table 1, we obtain the following estimate to these two real roots

$$\xi_{3,4} \approx \frac{-j_1 + |\bar{a}|(c + d_1 + j_1)^2 \pm \left\{ \left[j_1 - |\bar{a}|(c + d_1 + j_1) \right]^2 + 4(1 - |\bar{a}|)|\bar{a}| j_1 c \right\}^{1/2}}{2(1 - |\bar{a}|)} \tag{30}$$

$$\xi_3 > 0, \; \xi_4 < 0$$

which simplifies to eq. (26) and eq. (28) when the appropriate creep parameters are set equal to zero.

Finally, we consider the nonlinear Maxwell-Kelvin-Kelvin material, for which there are three roots in addition to the two previously discussed complex roots. If α_3 in Table 1 is negative, then for the corresponding values of $|\bar{a}|$ one can prove that α_4 must also be negative. However, if α_3 is positive, then α_4 may be either positive or negative. For these three cases there is still only one sign variation in the coefficient sequence, and thus there is again one and only one positive real root. The other two roots may in principle be either both negative real or a second complex conjugate pair, but our numerical calculations have yielded no characteristic frequency other than the one near the elastic frequency (see Section 4). The three real roots are estimated by the solutions to the cubic equation

$$\xi^3 + \left[j_1 + j_2 - \frac{(c+d_1+d_2)|\bar{a}|}{1-|\bar{a}|}\right]\xi^2 + \left[j_1 j_2 - \frac{(j_1 c + j_2 c + d_1 j_2 + d_2 j_1)|\bar{a}|}{1-|\bar{a}|}\right]\xi - \frac{j_1 j_2 c|\bar{a}|}{1-|\bar{a}|} = 0 \tag{31}$$

$$\xi_3 > 0 \quad \xi_4, \xi_5 < 0$$

An examination of the test function for Cardan's formula confirmed the observation that, for typical values of the parameters, the three solutions to eq. (31) are in fact real. Thus, the overall behaviours of the M-K-K, M-K, S-S (with $|\bar{a}| > \bar{a}_c$) and M materials are essentially the same, i.e. there is an initial period of decaying oscillation which is followed by a period of exponential growth.

For convenience, the character and the estimates of the roots for the various models are summarized in Table 2. The actual evaluation of the coefficients c_i in the solution [eq. (21)] from these roots is somewhat tedious for the more general models, and thus we will obtain the results for the M and S-S models only. Utilizing Table 2 we write the displacement solution for both these models as

$$v(x,t) \approx \sin m\pi x \left[(c_1 \cos \omega t + c_2 \sin \omega t)e^{-\beta t} + c_3 e^{\xi_3 t}\right] \tag{32}$$

Also, initial conditions (13 a-c) with (15) and (22) yield

$$v(x,0^+) = \sin m\pi x, \quad \frac{\partial v}{\partial t}(x,0^+) = 0, \quad \frac{\partial^2 v}{\partial t^2}(x,0^+) = -\omega^2 \sin m\pi x \tag{33}$$

Evaluating c_1, c_2, c_3 in eq. (32) from conditions (33), we obtain the complete solution for the M and S-S models as

TABLE 2 - Character of Roots for $|\bar{a}| < 1$ with Estimates

Material	No. of complex roots (complex conjugates with neg. real parts)	No. of pos. real roots	No. of neg. real roots
M	2 - Eq. (24)	1 - Eq. (26)	0
S-S ($|\tilde{a}| > \bar{a}_c$)	2 - Eq. (24)	1 - Eq. (28)	0
S-S ($|\tilde{a}| < \bar{a}_c$)	2 - Eq. (24)	0	1 - Eq. (28)
M-K	2 - Eq. (24)	1 - Eq. (30)	1 - Eq. (30)
M-K-K	2 - Eq. (24)	1 - Eq. (31)	2 - Eq. (31)

$$v(x,t) \approx \frac{\sin m\pi x}{(\xi_3+\beta)^2+\omega^2}\left\{\left[(\xi_3^2+2\beta\xi_3+\omega^2)\cos\omega t+\frac{\beta(\xi_3^2+\xi_3\beta+\omega^2)}{\omega}\sin\omega t\right]\cdot e^{-\beta t}+\beta^2 e^{\xi_3 t}\right\} \quad (34)$$

Note that although the coefficient c_3 is very small (since $\beta \ll \omega$) it is always positive, and thus for positive ξ_3 the displacement for t large is in the same direction as the initial displacement.

For linear viscoelastic materials it is well known that the S-S material is characterized by two critical loads, and that for materials with a long-time viscous behaviour (e.g. M, M-K, M-K-K) the lateral displacement always grows with time (e.g. see Distéfano, ref. 10). Accordingly, the results obtained here concerning the roots for a particular nonlinear material are consistent with previously established results from the theory of linear viscoelasticity. The roots which we have estimated in this section will be obtained by numerical computer calculations in the next section for a stainless steel beam at two levels of load and temperature.

4. NUMERICAL RESULTS

In order to illustrate the roots and solutions discussed in the previous section, we consider a 36 in. long stainless steel (type 316) beam with a rectangular cross-section which is 2 in. wide by 3 in. deep. For the axial force P_o and the temperature θ we consider two cases, i.e. $P_o = -39\,000$ lb with $\theta = 1300^oF$ and $P_o = -60,000$ lb with $\theta = 1500^oF$, where we shall soon see that the latter case satisfies $|\bar{a}| > \bar{a}_c$ [see eq. (27)] while the former satisfies $|\bar{a}| < \bar{a}_c$. The material properties ρ, n, λ, q_1, μ_1, τ_1, as listed in Table 3 (with lb, hr and in. units) are identical with those obtained in ref. 2 for a M-K model using the data of Garofalo et al.[11,12]. The values of E listed in Table 3 for the two temperatures were obtained by extrapolation from data at lower temperatures given in ref. 13, and they are slightly less than those used in ref. 2.

It remains for us to obtain some values for q_2, μ_2, τ_2, i.e. the creep parameters in the second Kelvin element of the M-K-K model. Reexamining the data in ref. 11, 12 we see that the relationship between the "initial" strain and the stress is clearly nonlinear and greater than the calculated linear elastic strain. We assume here that the difference between this initial strain and the calculated linear elastic strain is a time-dependent strain due to rapid transient creep. In the absence of more detailed data we set $q_2 = q_1$ and $\tau_2 = \tau_1 \times 10^{-4}$, and then calculate μ_2 from the data in ref. 11, 12 at stresses less than 10,000 psi. Although these estimated values are rather crude, they serve the purpose of enabling us to illustrate the effect of rapid transient creep. All material parameters for both temperatures are listed in Table 3.

TABLE 3 - Material Properties

	θ (°F)	
	1300	1500
ρ, 10^{-10} lb-hr^2/in^2	3.45	3.45
E, 10^7 lb/in^2	2.0	1.5
$n = q_1 = q_2$	3.64	3.50
λ, 10^4 lb/in^2-hr$^{1/n}$	19.18	7.86
μ_1, 10^4 lb/in^2	3.10	2.11
μ_2, 10^4 lb/in^2	7.45	5.26
$\tau_1 = \tau_2 \times 10^4$	333	21.7

TABLE 4 - Nondimensional Constants

	P_o = -30,000 lb θ^o = 1300°F	P_o = -60,000 lb θ^o = 1500°F
$\lvert\bar{a}\rvert$	4.377×10^{-2}	1.167×10^{-1}
$\bar{b}$	8.566×10^{-7}	1.142×10^{-6}
c	6.943×10^{-6}	1.071×10^{-3}
d_1	1.585×10^{-5}	4.925×10^{-3}
d_2	6.516×10^{-3}	2.013
$j_2 = j_1 \times 10^4$	8.342×10^{-3}	1.280×10^{-1}
$\bar{a}_c$	5.006×10^{-2}	2.592×10^{-3}

For the evaluation of the nondimensional material constants from eqs. (10) and (18) we select m=1 (first mode) and $T^* = 1/3600$ hr, which converts real time scale $T-T_0$ from hours to seconds [eq. (8)]. Material constants $|\bar{a}|$, $\bar{b}$, c, d_1, d_2, j_1, j_2 plus the critical load $\bar{a}_c$ [eq. (27)] are listed in Table 4 for the two previously mentioned cases of load and temperature. Note that for $P_o = -30,000$ lb at $\theta = 1300^oF$ we have $|\bar{a}| < \bar{a}_c < 1$, whereas for $P_o = -60,000$ lb at $\theta = 1500^oF$ we have $\bar{a}_c < |\bar{a}| < 1$. Thus, while both cases are below the critical linear elastic load as required by eq. (23), only the former case is also below the additional critical load for a standard solid material.

Using the nondimensional constants listed in Table 4, the roots to the characteristic equations given in Table 1 were obtained by a program on a high speed computer. The roots for $P_o = -30,000$ lb at $\theta = 1300^oF$ are presented in Table 5, and the roots for $P_o = -60,000$ lb at $\theta = 1500^oF$ are given in Table 6. When these numerically computed roots are compared with the simple estimates summarized in Table 2, we find that the estimates are remarkably accurate and in fact agree with the computer obtained roots to four significant figures. Finally, as an illustrative case we give below the complete numerical solution for the M model at $P_o = -60,000$ lb with $\theta = 1500^oF$ as obtained from eq. (34) and Table 6:

$$v(x,t) = \sin \pi x \{4.755 \times 10^{-13} \exp(1.416 \times 10^{-4} t) + \exp(-6.064 \times 10^{4} t) \cdot [\cos(879.4\, t) + 6.896 \times 10^{-7} \sin(879.4\, t)]\} . \qquad (35)$$

The deflection solutions based on the roots listed in Tables 5 and 6 are in complete conformity with the predictions of Section 3.2. As expected, all solutions contain two terms which oscillate at the elastic frequency ω [eq. (22)], although the $\sin \omega t$ term is negligible in the present example since the motion starts from rest [e.g. see eq. (35)]. As predicted by eq. (25) for β, the rate of decay of these oscillatory terms increases as the load and temperature increases and as additional creep components are added to the model. The material with the larger values of $|P_o|$ and θ (Table 6) and with the rapid transient creep terms included (i.e., M-K-K) exhibits sufficient damping ($\beta = 1.143$) to cause complete decay of the oscillatory terms over a period of several seconds.

Examining the real roots in Tables 5 and 6, we see that as predicted only the S-S material at the lower values of $|P_o|$ and θ (Table 5) does not have an associated positive real root. In this one case the perturbed displacement tends toward zero as $t \to \infty$, and thus this material under these conditions may be termed "stable". In the other seven cases the perturbed displacement tends toward infinity as $t \to \infty$, and accordingly these may be considered "unstable". One can show that the coefficient of the exponentially growing term is always positive as in eq. (35), and thus in these cases the column "collapses" (after a period of decaying oscillation) in the same direction as the initial displacement.

TABLE 5 - Roots for the Various Models at
$P_o = -30,000$ lb $\theta = 1300^oF$

Material	$\xi_{1,2} = -\beta \pm i\omega$		ξ_3	ξ_4	ξ_5
	β	ω			
E	-	1.057×10^3	-	-	-
M	3.630×10^{-6}	1.057×10^3	3.178×10^{-7}	-	-
S-S	8.283×10^{-6}	1.057×10^3	-1.085×10^{-7}	-	-
M-K	1.192×10^{-5}	1.057×10^3	6.301×10^{-7}	-4.208×10^{-7}	-
M-K-K	3.450×10^{-3}	1.057×10^3	6.628×10^{-7}	-8.117×10^{-3}	-4.148×10^{-7}

TABLE 6 - *Roots for the Various Models at*
$P_o = -60,000$ lb $\theta = 1500^oF$

Material	$\xi_{1,2} = -\beta \pm i\omega$		ξ_3	ξ_4	ξ_5
	β	ω			
E	-	8.794×10^2	-	-	-
M	6.064×10^{-4}	8.794×10^2	1.416×10^{-4}	-	-
S-S	2.788×10^{-3}	8.794×10^2	6.380×10^{-4}	-	-
M-K	3.394×10^{-3}	8.794×10^2	7.819×10^{-4}	-2.318×10^{-6}	-
M-K-K	1.143	8.794×10^2	1.396×10^{-1}	-7.371×10^{-4}	-2.254×10^{-6}

5. CONCLUDING REMARKS

We have demonstrated in detail how one may use simple hand calculations to construct the complete incremental displacement solution for a vibrating nonlinear viscoelastic beam under initial axial compression. Solutions were obtained for rather general constitutive relations, including those with the long-time viscous behaviour which is characteristic of metals at high temperatures. For such cases the lateral displacement always grows with time, even if the load is less than the critical load. The solutions obtained may be used to estimate a critical time at which the lateral displacement exceeds a prescribed acceptable value.

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APPENDIX - LIST OF SYMBOLS

a, b, c, d_i, j_i	nondimensional material constants, eqs. (10)
$\bar{a}, \bar{b}$	nondimensional material constants for mode m, eqs. (18)
$\bar{a}_c$	nondimensional critical load for standard solid material, eq. (27)
A	cross-sectional area of beam
c_i	constants of integration, eq. (21)
E	modulus of elasticity
ε	small quantity
$g(x)$	space dependence of nondimensional loading function, eq. (12)
I	second moment of area
L	length of beam
m	vibration mode number
$M(X, T)$	bending moment
n	creep power, eq. (1)
N	number of Kelvin elements
p	order of characteristic equation, Table 1
P_o	initial axial force on beam
P_{oc}	critical load for standard solid material, eq. (29)
q_i	creep power of i-th nonlinear Kelvin element, eq. (1)
t	nondimensional time, eqs. (8)
T	time
T^*	constant reference time, eqs. (8)
T_o	time at beginning of disturbance
$\bar{T}(t)$	function of time
v	nondimensional lateral displacement, eqs. (8)
V	lateral displacement
V^*	constant reference displacement, eqs. (8)
$W(X, T)$	lateral load on the beam
$w(x, t)$	nondimensional lateral loading, eqs. (8)
x	nondimensional axial coordinate, eqs. (8)
X	centroidal axis of beam
Y, Z	coordinates in cross-section of beam

<u>List of Symbols</u> (continued)

β	magnitude of real part of complex roots, eqs. (24) and (25)
ϵ	strain
$\eta(t)$	doublet function
θ	temperature
λ	steady creep parameter, eq. (1)
μ_i	transient creep parameter of i-th nonlinear Kelvin element, eq. (1)
ξ	root of characteristic equations, Table 1
ρ	linear density
σ	stress
τ_i	retardation time of i-th Kelvin element, eq. (1)
ω	nondimensional frequency, eq. (22)
$(\ )_o$	indicates initial value during interval $T < T_o$
$(\tilde{\ })$	indicates increment due to lateral disturbance
$(\dot{\ })$	indicates d/dt

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13. ABSTRACT

The free lateral vibration of a nonlinear viscoelastic beam-column subjected to an initial compressive axial load is considered. The constitutive law is formulated with a linear elastic term and with power functions of stress in the transient and steady creep terms, and is of the nonlinear generalized Kelvin type.

By assuming that the stress caused by the oscillation is of much smaller magnitude than the initial stress, the problem is linearized. The problem is analyzed for five special viscoelastic models using small deformation theory, and numerical results are discussed for a stainless steel alloy.

14. KEY WORDS

Nonlinear viscoelastic, lateral vibration, compressive load

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